MATH 1A – QUIZ 5 – SOLUTIONS

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(1) *(5 points, 1 point each)* In each question, find y'. Do **NOT** simplify your answers (unless otherwise specified)!

(a) $x^y = y^x$

The trick is to take $\ln$ of both sides:

$$\begin{aligned}
\ln(x^y) &= \ln(y^x) \\
y\ln(x) &= x\ln(y) \\
y'\ln(x) + y\frac{1}{x} &= \ln(y) + x\frac{y'}{y} \\
y'\ln(x) + \frac{y}{x} &= \ln(y) + \frac{xy'}{y} \\
y'\left(\ln(x) - \frac{x}{y}\right) &= \ln(y) - \frac{y}{x} \\
y' &= \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}
\end{aligned}$$

(b) $y = \sin^{-1}(\cos(x))$

$$y' = \frac{-\sin(x)}{\sqrt{1-\cos^2(x)}} = \frac{-\sin(x)}{\sqrt{\sin^2(x)}} = \frac{-\sin(x)}{|\sin(x)|}$$

Note: Careful! In general $\sqrt{a^2} = |a|$, **NOT** a!!!

Cool remark: When $0 < x < \pi$, then in fact $|\sin(x)| = \sin(x)$, so the answer becomes $y' = -1$. We'll see in section 4.9 that this implies $y = -x + C$, that is, $\sin^{-1}(\cos(x)) = -x + C$. Now plug in $x = \frac{\pi}{2}$ to get $\sin^{-1}(0) = -\frac{\pi}{2} + C$, so $0 = -\frac{\pi}{2} + C$, so $C = \frac{\pi}{2}$.

Combining, we get the following new cool identity:

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$$\boxed{\sin^{-1}(\cos(x)) = \frac{\pi}{2} - x} \qquad \text{(when } 0 < x < \pi\text{)}$$

(c) $y = (\ln(x))^{\ln(x)}$ (yo dawg...)

(1) $\ln(y) = \ln(x)\ln(\ln(x))$
(2) $\frac{y'}{y} = \frac{1}{x}\ln(\ln(x)) + \ln(x)\frac{1}{\ln(x)}\frac{1}{x} = \frac{\ln(\ln(x))+1}{x}$
(3) $y' = (\ln(x))^{\ln(x)}\left(\frac{\ln(\ln(x))+1}{x}\right)$

(d) $Pe^{yam} = x$ (P, a, m are constants. Write your answer without using y)

$$\begin{aligned}
Pe^{yam} &= x \\
Pe^{yam}(y'am) &= 1 \\
y'Pe^{yam}(am) &= 1 \\
y' &= \frac{1}{Pe^{yam}am} \\
y' &= \frac{1}{xam} \\
y' &= \frac{1}{max}
\end{aligned}$$

(And in fact maximization is a topic that we'll cover very soon :))

Other solution:

$$\begin{aligned}
Pe^{yam} &= x \\
\ln(Pe^{yam}) &= \ln(x) \\
\ln(P) + \ln(e^{yam}) &= \ln(x) \\
\ln(P) + yam &= \ln(x) \\
(\ln(P))' + (yam)' &= \frac{1}{x} \\
0 + y'(am) &= 1 \\
(am)y' &= \frac{1}{x} \\
y' &= \frac{1}{amx} \\
y' &= \frac{1}{max}
\end{aligned}$$

(e) The equation of the tangent line to the curve $y^2(y^2 - 4) = x^2(x^2 - 5)$ at the point $(0, -2)$

$$2yy'(y^2 - 4) + y^2(2yy') = 2x(x^2 - 5) + x^2(2x)$$

Now plug in $x = 0$, $y = -2$:

$$\begin{aligned}2(-2)y'((-2)^2-4)+(-2)^2(2(-2)y') =& 2(0)(0^2-5)+0^2(2)(0)\\ -4y'(0)-16y' =& 0\\ y' =& 0\end{aligned}$$

Hence the equation of the tangent line is $y-(-2)=0(x-0)$, so $\boxed{y=-2}$

(2) *(5 points)* Show that the sum of the $x-$ intercept and the $y-$ intercept of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is equal to c

Slope:

$$\begin{aligned}\frac{1}{2\sqrt{x}}+y'\left(\frac{1}{2\sqrt{y}}\right) =& 0\\ y'\left(\frac{1}{2\sqrt{y}}\right) =& -\frac{1}{2\sqrt{x}}\\ y' =& -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}\\ y' =& -\frac{2\sqrt{y}}{2\sqrt{x}}\\ y' =& -\frac{\sqrt{y}}{\sqrt{x}}\end{aligned}$$

Equation: At (x_0,y_0), the slope is $-\frac{\sqrt{y_0}}{\sqrt{x_0}}$, and so the equation of the tangent line at (x_0,y_0) is:

$$y-y_0=-\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$$

$y-$intercept:
To find the $y-$intercept, set $x=0$ and solve for y:

$$\begin{aligned}y-y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(0-x_0)\\ y-y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)\\ y-y_0 =& \sqrt{y_0}\sqrt{x_0}\\ y =& y_0+\sqrt{y_0}\sqrt{x_0}\end{aligned}$$

$x-$intercept:
To find the $x-$intercept, set $y=0$ and solve for x:

$$\begin{aligned}0 - y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)\\ -y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)\\ x - x_0 &= -\frac{\sqrt{x_0}}{\sqrt{y_0}}(-y_0)\\ x &= x_0 + \sqrt{x_0}\sqrt{y_0}\end{aligned}$$

Sum:
The sum of the $y-$ and $x-$ intercepts is:

$$(y_0 + \sqrt{y_0}\sqrt{x_0}) + (x_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0$$

But the trick is that:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0})^2 + 2\sqrt{x_0}\sqrt{y_0} + (\sqrt{y_0})^2 = (\sqrt{x_0} + \sqrt{y_0})^2$$

But since (x_0, y_0) is on the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$, we get $\sqrt{x_0} + \sqrt{y_0} = \sqrt{c}$.

And so, finally we get that the sum of the $x-$ and $y-$ intercepts is:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c$$